

# UNIVERSALLY KURATOWSKI-ULAM SPACE AND OPEN-OPEN GAMES

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**ABSTRACT.** We examine the class of spaces in which the second player has winning strategy in the open-open game. We shown that this space is not an universally Kuratowski-Ulam. We also show that the games  $G$  and  $G_7$  introduced by Daniels, Kunen, Zhou [2] are not equivalent.

## 1. INTRODUCTION

First we shall, recall some game introduced in [2] called  $G_2$ . Let  $X$  be a topological space equipped with a topology  $\mathcal{T}$  and let  $\mathcal{B} \subseteq \mathcal{T}$  be its base. The lenght of the game is  $\omega$ . Two players I and II take turns playing. At the  $n$ -th move II chooses a family  $\mathcal{P}_n$  consisting of open subset of  $X$  such that  $\text{cl} \bigcup \mathcal{P}_n = X$ , then I picks a  $V_n \in \mathcal{P}_n$ . I wins iff  $\text{cl} \bigcup_{n \in \omega} V_n = X$ . Otherwise II player wins. Denote by  $D_{cov}$  a collection of families  $\mathcal{F}$  consisting of open sets with  $\text{cl} \bigcup \mathcal{F} = X$ . We say that  $\sigma_{cov} : (\bigcup D_{cov})^{<\omega} \rightarrow D_{cov}$  is a *winning strategy for II player* in the game  $G_2$  if  $U_0, U_1, \dots$  is a sequence consisting of non-empty open subset with  $U_0 \in \sigma_{cov}(\emptyset) = \mathcal{P}_0 \in D_{cov}$  and  $U_n \in \sigma_{cov}(U_0, U_1, \dots, U_{n-1}) = \mathcal{P}_n \in D_{cov}$ , for all  $n \in \omega$ , then  $\text{cl} \bigcup_{n \in \omega} U_n \neq X$ .

In the paper [2] the authors introduced open-open game. We say that  $G$  is *open-open game* of length  $\omega$  if two players take turns playing; a round consists of player I choosing a non-empty open set  $U \subseteq X$  and player II choosing a non-empty open  $V \subseteq U$ ; I wins if the union of II's open sets is dense in  $X$ , otherwise II wins. Suppose that there exists a function

$$s_{op} : \bigcup \{\mathcal{B}^n : n \geq 0\} \rightarrow \mathcal{B}$$

such that for each sequence  $V_0, V_1, \dots$  consisting of non-empty elements of  $\mathcal{B}$  with  $s_{op}(V_0) \subseteq V_0$  and  $s_{op}(V_0, V_1, \dots, V_n) \subseteq V_n$ , for all  $n \in \omega$ ,

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then  $\text{cl} \bigcup_{n \in \omega} V_n \neq X$ . The function  $s_{op}$  is called a *winning strategy* for *II player* in the open-open game and we say that the space  $X$  is *II-favorable*.

It is known [2] that the open-open game  $G$  is equivalent to  $G_2$ . We consider only games with the length equal  $\omega$ . In [2] the authors introduced a game  $G_7$  which is played as follows: In the  $n$ -th inning II chooses  $\mathcal{O}_n$ , a family of open sets with  $\bigcup \mathcal{O}_n$  is dense in  $X$ . I responds with  $T_n$ , a finite subset of  $\mathcal{O}_n$ ; I wins if  $\bigcup_{n \in \omega} T_n$  is dense subset of  $X$ ; otherwise, II wins.

According to A. Szymański [13] a sequence  $\{\mathcal{P}_n : n \in \omega\}$  of open families in  $X$  is a *tiny sequence* if

- (1)  $\bigcup \mathcal{P}_n$  is dense in  $X$  for all  $n \in \omega$
- (2) if  $\mathcal{F}_n$  is a finite subfamily of  $\mathcal{P}_n$  for each  $n \in \omega$  then  $\bigcup \{\bigcup \mathcal{F}_n : n \in \omega\}$  is not dense in  $X$ .

We call a sequence  $\{\mathcal{P}_n : n \in \omega\}$  of open families in  $X$  a *1-tiny sequence* if

- (1)  $\bigcup \mathcal{P}_n$  is dense in  $X$  for all  $n \in \omega$
- (2) if  $F_n$  is a member of  $\mathcal{P}_n$  for each  $n \in \omega$  then  $\bigcup \{F_n : n \in \omega\}$  is not dense in  $X$ .

M. Scheepers used in the paper [12] negation of the existence of tiny sequence, and 1-tiny sequence and called this property  $S_{fin}(\mathcal{D}, \mathcal{D})$  and  $S_{fin}(\mathcal{D}, \mathcal{D})$  respectively. In this paper we use notion tiny sequence and 1-tiny sequence, because in some situation ( Theorem 1 and 2) we can define this sequence.

Recall another game  $G_4$  introduced in [2]. In the  $n$ -th inning player I chooses finite family  $\mathcal{A}_n$  of open subset. Player II responds with a finite subset  $\mathcal{B}_n$  of open subsets with  $|\mathcal{B}_n| = |\mathcal{A}_n|$  and for each  $V \in \mathcal{A}_n$  there exists  $W \in \mathcal{B}_n$  such that  $W \subseteq V$ . I wins if  $\bigcup_{n \in \omega} \bigcup \mathcal{B}_n$  is dense subset of  $X$ ; otherwise, II wins. One can prove in similar way as equivalence games  $G$  and  $G_2$  that the game  $G_7$  is equivalent to the game  $G_4$ .

Proofs of Theorems 1 and 2 one can find in [12, Th2, Th14]. We give new proofs of them, similar to the old, but our proofs shown directly how to construct a tiny sequence ( an 1-tiny sequence respectively) whenever there exists a winning strategy for II player in the game  $G_7$  ( $G_2$ ). From now we consider only in c.c.c. spaces.

**Theorem 1** (M. Scheepers, Theorem 2 [12] ). *II has winning strategy in the game  $G_7$  if, and only if, there exists a tiny sequence.*

**Theorem 2** (M. Scheepers, Theorem 14 [12]). *Player II has a winning strategy in the game  $G_2$  if, and only if, there exists an 1-tiny sequence.*

## 2. THE MAIN RESULTS

Recall that  $X$  is called *II-favorable* space if II player has winning strategy in the game  $G$ . If I Player has winning strategy in the game  $G$  then we say that the space is *I-favorable*.

The following theorem was proved by K. Kuratowski and S. Ulam, see [9]

*Let  $X$  and  $Y$  be topological spaces such that  $Y$  has countable  $\pi$ -weight. If  $E \subseteq X \times Y$  is a nowhere dense set, then there is  $P \subseteq X$  of first category such that the section  $E_x = \{y : (x, y) \in E\}$  is nowhere dense in  $Y$  for any point  $x \in X \setminus P$ .*

A space  $Y$  is *universally Kuratowski-Ulam\** (for short, *uK-U\** space), whenever for a topological space  $X$  and a nowhere dense set  $E \subseteq X \times Y$  the set

$$\{x \in X : \{y \in Y : (x, y) \in E\} \text{ is not nowhere dense in } Y\}$$

is meager in  $X$ , see D. Fremlin [6] (compare [3]). In the paper [7] authors shown that a compact I-favorable space is universally Kuratowski-Ulam and posed a question: *Does there exist a compact universally Kuratowski-Ulam space which is not I-favorable?* We will partial answer on this question namely we prove that II-favorable space is not universally Kuratowski-Ulam space.

**Theorem 3.** *Let  $X$  be a dense in itself space with  $\pi$ -base  $\mathcal{B} = \bigcup_{n \in \omega} \mathcal{B}_n$ , where  $\mathcal{B}_n$  is a maximal family of pairwise disjoint open sets and let  $Y$  be II-favorable c.c.c. space. Then the Kuratowski-Ulam theorem does not hold in  $X \times Y$ .*

*Proof.* By Theorem 2 there is an 1-tiny sequence  $\{\mathcal{P}_n : n \in \omega\}$ . Since the space  $X$  satisfies c.c.c. we can assume that each  $\mathcal{P}_{n+1}$  is countable family of open subsets pairwise disjoint. We can also assume that every  $\mathcal{P}_{n+1}$  is refinement of  $\mathcal{P}_n$  i.e. each member of  $\mathcal{P}_{n+1}$  is a subset of member of  $\mathcal{P}_n$ . Let  $\{V_\sigma^n : \sigma \in {}^n\mathbb{N}\}$  be an enumeration of the family  $\mathcal{P}_n$ .

We can assume that  $\mathcal{B}_{n+1}$  is refinement of  $\mathcal{B}_n$  and  $|\{V \in \mathcal{B}_{n+1} : V \subseteq U\}| \geq \omega$  for each  $U \in \mathcal{B}_n$ . For each  $n \in \mathbb{N}$  fix a function  $f_n : \mathcal{B}_n \rightarrow {}^n\mathbb{N}$  such that

$$(1) \{f_{n+1}(V) : V \in \mathcal{B}_{n+1} \& V \subseteq U\} = {}^{n+1}\mathbb{N} \text{ for every } n \in \mathbb{N} \text{ and } U \in \mathcal{B}_n,$$

(2) if  $U \subset V$  then  $f_{n+1}(U) \supset f_n(V)$  for every  $U \in \mathcal{B}_{n+1}$  and  $V \in \mathcal{B}_n$

Consider an open set

$$E = \bigcup \{ \bigcup \{ U \times V_{f_n(U)}^n : U \in \mathcal{B}_n \} : n \in \mathbb{N} \}.$$

We shall show that  $E$  is dense and  $E_x = \{y \in Y : (x, y) \in E\}$  is not dense for each  $x \in X$ . If  $x \in X \setminus \bigcap \{ \bigcup \mathcal{B}_n : n \in \mathbb{N} \}$  then it is easy to see that  $E_x$  is not dense. If  $x \in \bigcap \{ \bigcup \mathcal{B}_n : n \in \mathbb{N} \}$  then by condition (2) there is  $\sigma \in {}^\mathbb{N}\mathbb{N}$  such that for each  $n \in \mathbb{N}$  there exists  $U_n \in \mathcal{B}_n$  with  $f_n(U_n) = \sigma|n$  and  $x \in \bigcap \{ U_n : n \in \mathbb{N} \}$ , hence  $E_x = \{x\} \times \bigcup \{ V_{\sigma|n}^n : n \in \mathbb{N} \}$ . Since  $V_{\sigma|n}^n \in \mathcal{P}_n$  for each  $n \in \mathbb{N}$  and  $\{\mathcal{P}_n : n \in \omega\}$  is 1-tiny sequence the set  $\bigcup \{ V_{\sigma|n}^n : n \in \mathbb{N} \}$  is not dense.

Now we show that  $E$  is dense set. Let  $W \times U$  be any open set. Since  $\mathcal{B}$  is  $\pi$ -base there are  $n \in \mathbb{N}$  and  $U_0 \in \mathcal{B}_n$  such that  $U_0 \subseteq W$ . By (1) and  $\mathcal{P}_{n+1}$  is dense family, we get  $\bigcup \{ U_0 \times V_{f_n(U_0)}^n : n \in \mathbb{N} \} \cap W \times U \neq \emptyset$   $\square$

Since  $\mathbb{R}$  with natural topology satisfies assumption of the above Theorem and every universally Kuratowski-Ulam space is c.c.c. space we get the following Theorem.

**Theorem 4.** *II-favorable space is not universally Kuratowski-Ulam space.*

Following to [10, p.86 - 91] recall category measure space. If  $X$  is a topological space with finite measure  $\mu$  defined on the  $\sigma$ -algebra  $S$  of sets having the property of Baire, and if  $\mu(E) = 0$  if and only if  $E$  is of first category, then  $(X, S, \mu)$  is called a *category measure space*. An example of a regular Baire space which is category measure space, is open interval  $(0, 1)$  with Lebesgue measure  $\mu_l$  and density topology  $\mathcal{T}_d$ , see [10]. For density topology and measurable set  $A \subseteq (0, 1)$  the following conditions are equivalent:

- (1)  $\mu_l(A) = 0$ ,
- (2)  $A$  is closed and nowhere dense .

In the space  $((0, 1), \mathcal{T}_d)$  there is 1-tiny sequence but there is not tiny sequence. Indeed define 1-tiny sequence in the following way: let  $\mathcal{P}_n = \{U : U \in \mathcal{T}_d \text{ and } \mu_l(U) \leq \frac{1}{3^n}\}$ . If  $\{U_n : n \in \mathbb{N}\}$  is a family choosing by I player then  $\mu_l(\bigcup \{U_n : n \in \mathbb{N}\}) \leq \frac{1}{2}$ . Therefore  $\{U_n : n \in \mathbb{N}\}$  is not dense family. Now assume that there exists tiny sequence  $\{\mathcal{P}_n : n \in \mathbb{N}\}$ . In each stage we choose finite subfamily  $\mathcal{R}_n \subset \mathcal{P}_n$  such that  $\mu_l(\bigcup \{ \bigcup \mathcal{R}_i : i \leq n \}) \geq 1 - \frac{1}{n}$ , hence we get dense family  $\bigcup \{ \mathcal{R}_n : n \in \mathbb{N} \}$ .

The authors of the paper [2] ask ( Question 4.3 ) if a player has a winning strategy in the game  $G$  if, and only if, the same player has a winning strategy in the game  $G_7$ . The author of paper [12] shown that if  $\text{cov}(\mathcal{M}) < \mathfrak{d}$  the answer is NO. We show that games  $G$  and  $G_7$  are not equivalent.

**Corollary 5.** *The game  $G$  is not equivalent to the game  $G_7$*

*Proof.* By Theorem 2 a winnig strategy of II player in the game  $G$  is equivalent with existence of 1-tiny sequence and by Theorem 1 winnig strategy of player II in the game  $G_7$  is equivalent with existence of tiny sequence. Since in the space  $((0, 1), \mathcal{T}_d)$  there is an 1-tiny sequence but there is not tiny sequence we get that games  $G$  and  $G_7$  are not equivalent.  $\square$

Since the game  $G_7$  is equivalent to the game  $G_4$ , we get the following

**Corollary 6.** *The game  $G$  is not equivalent to the game  $G_4$*

### 3. PROOFS OF THEOREM 1 AND THEOREM 2

*Proof of Theorem 1.* We show only that if II has winning strategy in the game  $G_7$  then, we can construct tiny sequence. Let  $s$  be strategy for player II. Define a family  $(U_\tau : \tau \in {}^{<\omega}\mathbb{N})$  as follows  $U_\emptyset = \emptyset$ .  $(U_n : n \in \mathbb{N})$  enumerates a family  $s(\emptyset)$ , and  $\{U_n : n \in \mathbb{N}\}$  is an increasing sequence of open set with  $\bigcup \{U_n : n \in \mathbb{N}\}$  dense in  $X$ . For each  $n_1$  a family  $\{U_{n_1, n} : n \in \mathbb{N}\}$  enumerates a family  $s(U_{n_1})$ , in increasing order. The family  $(U_\tau : \tau \in {}^{<\omega}\mathbb{N})$  has the following properties for each  $\sigma$

- (1) if  $m < n$  then  $U_{\sigma \smallfrown m} \subseteq U_{\sigma \smallfrown n}$
- (2) for all  $n$   $U_\sigma \subseteq U_{\sigma \smallfrown n}$
- (3)  $\{U_{\sigma \smallfrown n} : n \in \mathbb{N}\}$  is dense family in  $X$

Define for each  $n$  and  $k$  a set

$$U_k^n = \begin{cases} U_k & \text{if } n = 1 \\ \bigcap \{U_{\sigma \smallfrown k} : \sigma \in {}^{n-1}\mathbb{N}\} \cap U_k^{n-1} & \text{otherwise} \end{cases}$$

The family  $\mathcal{U}_n = \{U_k^n : k \in \mathbb{N}\}$  is an ascending chain of open sets. Next we show by induction on  $n$  that each  $\mathcal{U}_n$  is dense family. The family  $\mathcal{U}_1 = \{U_n : n \in \mathbb{N}\}$  is dense family by property (3). Assume that  $\mathcal{U}_i$  are dense family for  $i \leq n$  and we will show that  $\mathcal{U}_{n+1}$  is dense family. Let  $U$  be non-empty open subset. There is  $k_1 \in \mathbb{N}$  with  $V_1 = U \cap U_{k_1}^n \neq \emptyset$ . There are only finitely many function  $\sigma : n \rightarrow k_1 + 1$  let

$\{\sigma_i : i \leq (k_1 + 1)^n\}$  be enumeration of  ${}^n(k_1 + 1)$ . Since  $\{U_{\sigma_1 \frown n} : n \in \mathbb{N}\}$  is dense family there exists  $k_2 \in \mathbb{N}$  such that  $V_2 = V_1 \cap U_{\sigma_1 \frown k_2} \neq \emptyset$  and so on. Let  $k_0 = \max\{k_1, \dots, k_{k_1+1}\}$  and  $p = (k_1 + 1)^n + 1$ , then

$$V_p \subseteq U_{\sigma \frown k_0} \text{ for all } \sigma \in {}^n(k_1 + 1).$$

Observe that

$$V_p \subseteq V_1 \subseteq U_{k_1}^n \subseteq U_{\sigma \frown k_1} \subseteq U_{\sigma \frown k \frown k_0}$$

for all  $\sigma \in {}^{n-1}\mathbb{N}$  and  $k \geq k_1$ . Fix  $\sigma \in {}^n\mathbb{N} \setminus {}^n(k_1 + 1)$  and let  $j \leq n$  be minimal number such that  $\sigma(j) > k_1$ . Then

$$V_p \subseteq U_{k_1}^n \subseteq \dots \subseteq U_{k_1}^j \subseteq U_{\sigma|(j-1) \frown k_1} \subseteq U_{\sigma|j} \subseteq U_\sigma \subseteq U_{\sigma \frown k_0}$$

Hence we get  $V_p \subseteq U_{\sigma \frown k_0}$  for all  $\sigma \in {}^n\mathbb{N}$  and finally  $V_p \subseteq U_{k_0}^{n+1} \cap U$ .

We will show that  $\{\mathcal{U}_n : n \in \mathbb{N}\}$  is tiny sequence. Since each  $\mathcal{U}_n$  is an ascending chain we can assume that we choose for each  $n$  an element  $U_{k_n}^n \in \mathcal{U}_n$ . Finally observe that for each  $n$  we get  $U_{k_n}^n \subseteq U_{k_1, \dots, k_n}$ . The sequence of moves  $U_{k_1}, U_{k_1, k_2}, \dots$  of Two has property  $\text{cl} \bigcup \{U_{k_1}, U_{k_1, k_2}, \dots\} \neq X$ , hence  $\text{cl} \bigcup \{U_{k_n}^n : n \in \mathbb{N}\} \subseteq \text{cl} \bigcup \{U_{k_1}, U_{k_1, k_2}, \dots\} \neq X$ .  $\square$

Similarly to [11] we introduce the following game  $G_7^\sigma$ : in the  $n$ -th inning II chooses  $\mathcal{O}_n$ , family of open sets with dense union. I responds with  $T_n$ , a finite subset of  $\mathcal{O}_n$ ; player I wins if  $\bigcap_{n \in \omega} \bigcup_{m > n} (\bigcup T_m)$  is dense subset of  $X$ ; otherwise, II wins. It's clear that if I has a winning strategy in  $G_7^\sigma$  then I has winning strategy in  $G_7$ , and if II has winning strategy in  $G_7$  then II has winning strategy in  $G_7^\sigma$ .

We say that a sequence  $\{A_n : n \in \omega\}$  of open families with dense union is a *weak tiny sequence* if for every sequence  $\{E_n : n \in \omega\}$  with  $E_n$  is finite subset of  $A_n$  a set  $\bigcap_{k \in \omega} \bigcup_{n \geq k} \bigcup E_n$  is not dense in  $X$ . It is clear that if there exists tiny sequence then there exists weak tiny sequence.

**Fact 1.** *Let  $X$  be Baire space with ccc and let  $\{A_n : n \in \omega\}$  be a open family with dense union in  $X$ . If the family  $\{A_n : n \in \omega\}$  is weak tiny sequence then there exists  $k \in \omega$  such that  $\{A_n : n \geq k\}$  is tiny sequence.*

*Proof.* Fix a sequence  $\{A_n : n \in \omega\} \subseteq \mathcal{D}$  of open families with dense union and assume that each  $\{A_n : n \geq k\}$  is not tiny sequence. Then for each  $k$  there exists a sequence  $\{E_n^k : n \in \omega\}$  such that  $E_n^k$  is finite subset of  $A_n$  and  $\bigcup_{k \leq n} \bigcup E_n^k$  is dense in  $X$ . Let  $E'_n = \bigcup_{k \leq n} E_n^k$ . Each  $E'_n$  is finite subset of  $A_n$  and for every  $k \in \omega$  we have  $X = \text{cl} \bigcup_{n \geq k} \bigcup E_n^k \subseteq$

$\text{cl} \bigcup_{n \geq k} \bigcup E'_n$ ; this contradicts with property  $\{A_n : n \in \omega\}$  is weak tiny sequence.  $\square$

The same proof as proof of the Theorem 1 works when we do a swap the game  $G_7$  to the game  $G_7^\sigma$  and the tiny sequence to the weak tiny sequence, hence we get the following Theorem.

**Theorem 7.** *Player II has a winning strategy in the game  $G_7^\sigma$  if, and only if, there exists a weak tiny sequence.*

*Proof of the Theorem 2.* We proof only one direction; if II has a winning strategy in the game  $G_2$  then, we can construct an 1-tiny sequence. Let  $s$  be a winning strategy for player II. Without lost of the generality we can assume that strategy  $s$  is defined on the collection of families  $\mathcal{A}$  of open subsets with  $|\mathcal{A}| = \omega$  and  $\text{cl} \bigcup \mathcal{A} = X$ . Let  $\{U_n : n \in \mathbb{N}\}$  enumerates first move  $s(\emptyset)$  For each  $n_1$  a family  $\{U_{n_1, n} : n \in \mathbb{N}\}$  enumerates a family  $s(U_{n_1})$ , and so on. The family  $\{U_\tau : \tau \in {}^{<\omega}\mathbb{N}\}$  has the following property: for each  $\sigma$  the family  $\{U_{\sigma \frown n} : n \in \mathbb{N}\}$  is dense in  $X$ . For fixed  $m$  and  $j \in \mathbb{N}$  and function  $\rho : j^m \rightarrow \mathbb{N}$ , define the set

$$U_\rho(m, j) := \bigcap_{\sigma \in j^m} \left( \bigcup \{U_{\sigma \frown \rho \upharpoonright i} : i \leq j^m\} \right)$$

and then for fixed  $m$  and  $j$  define

$$\mathcal{U}(m, j) := \{U_\rho(m, j) : \rho : j^m \rightarrow \mathbb{N}\}$$

Observe that each family  $\mathcal{U}(m, j)$  is dense in  $X$ . Indeed, if  $U$  is non-empty open set, then  $\{\sigma_i : i \leq j^m\}$  enumerates all function  $\sigma \in j^m$ . Since  $\{U_{\sigma \frown n} : n \in \mathbb{N}\}$  is dense in  $X$ , there is  $i_1 \in \mathbb{N}$  such that  $V_1 = U \cap U_{\sigma \frown i_1} \neq \emptyset$ . We can find  $i_2 \in \mathbb{N}$  such that  $V_2 = V_1 \cap U_{\sigma \frown i_1 \frown i_2} \neq \emptyset$ , and so on. Then there is  $\rho(k) = i_k$  where  $k \leq j^m$  such that  $U \cap U_\rho(m, j) \neq \emptyset$

Now we describe some function  $c$ , if it will be a winning strategy for  $2^{nd}$  player in the game  $G_7^\sigma$  then by the previous Theorem 1 and Fact1 we can construct tiny sequence, hence 1-tiny sequence. In other case we say what to do.

For a first move  $2^{nd}$  puts  $j_1 = m_1 = 1$  and plays  $c(\emptyset) = \mathcal{U}(m_1, j_1)$ . For a response  $T_1 \subset \mathcal{U}(m_1, j_1)$  by  $1^{st}$ ,  $2^{nd}$  puts

$$m_2 = m_1 + j_1^{m_1},$$

and  $j_2 > j_1$  is at least the maximum of all values of  $\sigma$ 's for which  $U_\sigma(m_1, j_1)$  is in  $T_1$  i.e.  $j_2 > a_1 = \max \bigcup \{\text{rang}(\sigma) : U_\sigma(m_1, j_1) \in T_1\}$ .

Then  $2^{nd}$  plays  $c(T_1) = \mathcal{U}(m_2, j_2)$ . For a response  $T_2 \subset \mathcal{U}(m_2, j_2)$  by  $1^{st}$ ,  $2^{nd}$  puts

$$m_3 = m_2 + j_2^{m_2},$$

and

$$j_3 > j_2, a_2 = \max \bigcup \{\text{rang}(\sigma) : U_\sigma(m_2, j_2) \in T_2\}.$$

Then  $2^{nd}$  plays  $c(T_1, T_2) = \mathcal{U}(m_3, j_3)$ , and so on. If the function  $c$  is not the winning strategy for  $2^{nd}$  player, look at a play  $c(\emptyset), T_1, c(T_1), T_2, c(T_1, T_2), T_3 \dots$  which is lost by  $2^{nd}$  player. Then  $\bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} (\bigcup T_k)$  is dense subset of  $X$  and there are increasing sequence  $(j_n : n \in \mathbb{N})$  and  $(m_n : n \in \mathbb{N})$  such that for each  $n$ :

- (1)  $m_{n+1} = m_n + j_n^{m_n}$
- (2)  $c(T_1, \dots, T_n) = \mathcal{U}(m_{n+1}, j_{n+1})$
- (3)  $j_{n+1} > j_n, a_n = \max \bigcup \{\text{rang}(\sigma) : U_\sigma(m_n, j_n) \in T_n\}.$

For fixed sequences  $(j_n : n \in \mathbb{N})$  and  $(m_n : n \in \mathbb{N})$  define for each  $n$  a family  $\mathcal{W}_n$  consists of all subset of the form:

$$W(k_1, \dots, k_n; \sigma_1, \dots, \sigma_n) = \bigcap_{i \leq n} U_{\sigma_i}(m_{k_i}, j_{k_i})$$

where  $k_1 < k_2 < \dots < k_n$  and  $U_{\sigma_i}(m_{k_i}, j_{k_i}) \in T_{k_i}$  and  $\sigma_i : (j_{k_i})^{m_{k_i}} \rightarrow j_{k_i+1}$ .

Each  $\mathcal{W}_n$  is a family of open subsets with dense union. Let  $U$  be open non-empty set. Since  $\bigcup_{k \in \mathbb{N}} (\bigcup T_k)$  is dense subset of  $X$  then there exists  $k_0$  and  $U_{\sigma_0}(m_{k_0}, j_{k_0}) \in T_{k_0}$  such that  $V_1 = U \cap U_{\sigma_0}(m_{k_0}, j_{k_0}) \neq \emptyset$ . Since  $\bigcup_{k \geq k_0+1} (\bigcup T_k)$  is dense subset of  $X$  there exists  $k_1 > k_0$  and  $U_{\sigma_1}(m_{k_1}, j_{k_1}) \in T_{k_1}$  such that  $V_2 = V_1 \cap U_{\sigma_1}(m_{k_1}, j_{k_1}) \neq \emptyset$  and so on. We prove that  $\{\mathcal{W}_n : n \in \omega\}$  is an 1-tiny sequence. Let  $\{W(k_1^n, \dots, k_n^n; \sigma_1^n, \dots, \sigma_n^n) : n \in \omega\}$  be any sequence such that  $W(k_1^n, \dots, k_n^n; \sigma_1^n, \dots, \sigma_n^n) \in \mathcal{W}_n$ . Observe that for each  $n$  and  $i \in \{1, \dots, n\}$  we have

$$(*) \ W(k_1^n, \dots, k_n^n; \sigma_1^n, \dots, \sigma_n^n) \subseteq U_{\sigma_i^n}(m_a, j_a) \subseteq U_{\sigma \restriction \sigma_i^n|1} \cup U_{\sigma \restriction \sigma_i^n|2} \cup \dots \cup U_{\sigma \restriction \sigma_i^n|(j_a)^{m_a}},$$

where  $a = k_i^n$  and  $\sigma \in {}^{m_a}j_a$ . Recursively we choose a sequence  $\{s_n : n \in \mathbb{N}\}$  such that  $s_1 = k_1^1$ ,  $s_2 \in \{k_1^2, k_2^2\} \setminus \{s_1\}$  and  $s_n \in \{k_1^n, \dots, k_n^n\} \setminus \{s_1, \dots, s_{n-1}\}$ , it is possible because  $k_1^n < \dots < k_n^n$ , and  $|\{k_1^n, \dots, k_n^n\}| = n$ . The sequence  $\{s_n : n \in \mathbb{N}\}$  consists of pairwise different element, so we can enumerate it as a strictly increasing



sequence  $\{a_n : n \in \mathbb{N}\}$ . Now we define a function  $f : \mathbb{N} \rightarrow \mathbb{N}$ . Take first a number  $a_1 = s_{n_1} = k_{i_1}^{n_1}$  for some  $i_1 \in \{1, \dots, n_1\}$ , and choose any function  $\sigma_0 \in {}^{m_{a_1}}j_{a_1}$  and define

$$f_0(i) = \sigma_0(i) \text{ for } i \in m_{a_1},$$

then we get

$$\begin{aligned} W(k_1^{n_1}, \dots, k_{n_1}^{n_1}; \sigma_1^{n_1}, \dots, \sigma_{n_1}^{n_1}) &\subseteq U_{\sigma_{i_1}^{n_1}}(m_{a_1}, j_{a_1}) \subseteq \\ &U_{f_0 \frown \sigma_{i_1}^{n_1}|1} \cup U_{f_0 \frown \sigma_{i_1}^{n_1}|2} \cup \dots \cup U_{f_0 \frown \sigma_{i_1}^{n_1}|j_{a_1}^{m_{a_1}}} \end{aligned}$$

Next define

$$f_1(i) = \sigma_{i_1}^{n_1}(i) \text{ for } i \in \{m_{a_1}, \dots, m_{a_1} + j_{a_1}^{m_{a_1}} - 1\}.$$

Recall that  $m_{a_1} + j_{a_1}^{m_{a_1}} = m_{a_1+1}$ . If  $a_1 + 1 < a_2$  then the set  $m_{a_2} \setminus m_{a_1+1} = \{m_{a_1+1}, \dots, m_{a_2} - 1\}$  is non-empty and we define

$$f_1(i) = 0 \text{ for } i \in m_{a_2} \setminus m_{a_1+1} = \{m_{a_1+1}, \dots, m_{a_2} - 1\}$$

Therefore we have defined a function  $f_0 \frown f_1 : m_{a_2} \rightarrow j_{a_2}$  and we get

$$\begin{aligned} W(k_1^{n_2}, \dots, k_{n_2}^{n_2}; \sigma_1^{n_2}, \dots, \sigma_{n_2}^{n_2}) &\subseteq U_{\sigma_{i_2}^{n_2}}(m_{a_2}, j_{a_2}) \subseteq \\ &U_{f_0 \frown f_1 \frown \sigma_{i_2}^{n_2}|1} \cup U_{f_0 \frown f_1 \frown \sigma_{i_2}^{n_2}|2} \cup \dots \cup U_{f_0 \frown f_1 \frown \sigma_{i_2}^{n_2}|j_{a_2}^{m_{a_2}}} \end{aligned}$$

Take next number  $a_2 = k_{i_2}^{n_2}$  and define

$$f_2(i) = \sigma_{i_2}^{n_2}(i) \text{ for } i \in \{m_{a_2}, \dots, m_{a_2} + j_{a_2}^{m_{a_2}} - 1\},$$

If  $a_2 + 1 < a_3$  then the set  $m_{a_3} \setminus m_{a_2+1} = \{m_{a_2+1}, \dots, m_{a_3} - 1\}$  is nonempty and we define

$$f_2(i) = 0 \text{ for } i \in m_{a_3} \setminus m_{a_2+1} = \{m_{a_2+1}, \dots, m_{a_3} - 1\}$$

Therefore we have defined the function  $f_1 \frown f_2 : (m_{a_2} + j_{a_2}^{m_{a_2}}) \rightarrow j_{a_2+1}$ , and so on. Let  $f = f_0 \frown f_1 \frown f_2 \frown \dots$ . By construction of function  $f$  and property (\*) we get  $\bigcup \{W(k_1^n, \dots, k_n^n; \sigma_1^n, \dots, \sigma_n^n) : n \in \omega\} \subseteq \bigcup \{U_{f|n} : n \in \mathbb{N}\}$ . The sequence  $\{U_{f|n} : n \in \mathbb{N}\}$  is a respond of player I in the game  $s(\emptyset), U_{f(1)}, s(U_{f(1)}), U_{f(1), f(2)}, s(U_{f(1)}, U_{f(1), f(2)}), \dots$ . Therefore  $\text{cl} \bigcup \{W(k_1^n, \dots, k_n^n; \sigma_1^n, \dots, \sigma_n^n) : n \in \omega\} \subseteq \text{cl} \bigcup \{U_{f|n} : n \in \mathbb{N}\} \neq X$ . This complete the proof.  $\square$

#### 4. SOME REMARKS

It is known that on the  $\omega_1$  with discrete topology II player has winning strategy in the game  $G_7$ , but one can posed a question:

*Is it possible to construct a tiny sequence  $\{\mathcal{P}_n\}_{n \in \omega}$  on discrete space of the size  $\omega_1$  with  $|\mathcal{P}_n| = \omega$  for all  $n \in \omega$  ?*

The following Remark 1 gives the answer, it is possible if and only if dominating number is equal  $\omega_1$ . This is reformulation of well know results about critical cardinal number, see W. Just, A. W. Miller, M. Scheepers and P. J. Szeptycki [5] and D. Fremlin, A. W. Miller [4] and B. Tsaban [14].

Recall that a family  $\mathcal{R} \subseteq {}^\omega\omega$  is a *dominating* family if for each  $f \in {}^\omega\omega$  there is  $g \in \mathcal{R}$  such that  $f \leq^* g$ . The *dominating number*  $\mathfrak{d}$  is the smallest cardinality of dominating family:

$$\mathfrak{d} = \min\{|\mathcal{R}| : \mathcal{R} \text{ is dominating}\}.$$

**Remark 1.** *The smallest cardinality  $\kappa$  such that there exists tiny sequence  $\{\mathcal{P}_n\}_{n \in \omega}$  on the discrete space of the size  $\kappa$  with  $|\mathcal{P}_n| = \omega$  for all  $n \in \omega$  is equal  $\mathfrak{d}$*

Recall a definition of Baire number  $\text{cov}(\mathcal{M})$  for the ideal  $\mathcal{M}$  of meager subsets of real line  $\mathbb{R}$ :

$$\text{cov}(\mathcal{M}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{M} \text{ and } \bigcup \mathcal{A} = \mathbb{R}\}.$$

T. Bartoszyński [1] proved that  $\text{cov}(\mathcal{M})$  is the cardinality of the smallest family  $\mathcal{F} \subseteq {}^\omega\omega$  such that

$$\forall (g \in {}^\omega\omega) \exists (f \in \mathcal{F}) \forall (n \in \omega) f(n) \neq g(n).$$

We get another well known characterization of such families by 1-tiny sequence.

**Remark 2.** *The smallest cardinality  $\kappa$  such that there exists 1-tiny sequence  $\{\mathcal{P}_n\}_{n \in \omega}$  on the discrete space of the size  $\kappa$  with  $|\mathcal{P}_n| = \omega$  for all  $n \in \omega$  is equal  $\text{cov}(\mathcal{M})$ .*

We give the proof for the sake of completeness. We shall prove that the smallest cardinality of family  $\mathcal{F} \subseteq {}^\omega\omega$  such that

$$(\text{dif}) \quad \forall (g \in {}^\omega\omega) \exists (f \in \mathcal{F}) \forall (n \in \omega) f(n) \neq g(n)$$

is equal the smallest cardinality  $\kappa$  such that there exists 1-tiny sequence  $\{\mathcal{P}_n\}_{n \in \omega}$  on the discrete space  $\kappa$  with  $|\mathcal{P}_n| = \omega$  for all  $n \in \omega$ .

*Proof.* Let  $\mathcal{F} = \{f_\alpha : \alpha < \kappa\} \subseteq {}^\omega\omega$  be a family with property (dif). Define  $A_n^i = \{f \in \mathcal{F} : f(i) = n\}$  for every  $i, n \in \omega$ . Let  $\mathcal{P}_i = \{A_n^i : n \in \omega\}$ . We will show that  $\{\mathcal{P}_i\}_{i \in \omega}$  is a 1-tiny sequence. Assume that we have chosen  $A_{n_i}^i \in \mathcal{P}_i$  for each  $i \in \omega$ . Define a function  $g(i) = n_i$ . Since  $\mathcal{F}$  has property (dif) there is  $f \in \mathcal{F}$  such that  $\forall(i \in \omega) f(i) \neq g(i)$ . Therefore we get  $f \in \mathcal{F} \setminus \bigcup \{A_{n_i}^i : i \in \omega\}$

Let  $\{\mathcal{P}_n\}_{n \in \omega}$  be an 1-tiny sequence with  $|\mathcal{P}_n| = \omega$  and  $\bigcup \mathcal{P}_n = \kappa$  for each  $n \in \omega$ . We can assume that each  $\mathcal{P}_n$  consists of pairwise disjoint subset of  $\kappa$ . Let  $\{A_k^n : k \in \omega\}$  be an enumeration of  $\mathcal{P}_n$ . We define a function  $f_x \in {}^\omega\omega$  for each  $x \in \kappa$  in the following way  $f_x(i) = n$  where  $x \in A_n^i$  for each  $i \in \omega$ . The family  $\{f_x : x \in \kappa\}$  has property (dif). Indeed, let  $g \in {}^\omega\omega$  be any function. Since  $\{\mathcal{P}_n\}_{n \in \omega}$  is 1-tiny sequence, choose  $x \in \kappa \setminus \bigcup \{A_{g(i)}^i : i \in \omega\}$ . Finally, observe that  $f_x(i) \neq g(i)$  for every  $i \in \omega$ .  $\square$

We shall recall definition of the bounding number

$$\mathfrak{b} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq {}^\omega\omega \text{ and } \forall(g \in {}^\omega\omega) \exists(f \in \mathcal{F}) \neg(f \leq^* g)\}$$

We say that a sequence  $\{\mathcal{P}_n\}_{n \in \omega}$  of open families in  $X$  is a **b-tiny sequence** if

- (1)  $\bigcup \mathcal{P}_n$  is dense in  $X$  for all  $n \in \omega$
- (2) if  $\mathcal{F}_n$  is a finite subfamily of  $\mathcal{P}_n$  for each  $n \in \omega$  then there exists sequence  $\{n_i : i \in \omega\}$  such that  $\bigcup \{\bigcup \mathcal{F}_{n_i} : i \in \omega\}$  is not dense in  $X$ .

We get the next reformulation of the bounding number.

**Remark 3.** *The smallest cardinality  $\kappa$  such that there exists **b-tiny** sequence  $\{\mathcal{P}_n\}_{n \in \omega}$  on the discrete space of the size  $\kappa$  with  $|\mathcal{P}_n| = \omega$  for all  $n \in \omega$  is equal  $\mathfrak{b}$ .*

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